1. geometric series $a + ar + ar^2 + ar^{n-1} + \cdots$ 几何級數

$$\sum_{n=1}^{\infty} a r^{n-1} = \frac{a}{1-r} \quad \text{if } |r|<1$$

遞 遞

2. telescoping series $(a_1 - a_2) + (a_2 - a_3) + \cdots$

2.1 $S_n = a_1 - a_{n+1}$ $\Rightarrow$ nth partial sum

2.2 if $\lim_{n\to\infty} a_n = L \Rightarrow S = a_1 - L$

3. Convergent Series

if $\sum_{n=1}^{\infty} a_n$ Converges, the $\lim_{n\to\infty} a_n = 0$

$\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges

4. Theorem 5.15

$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e \Rightarrow \left(1+\frac{1}{n}\right)^n \to e$

9.1	1C	2D	3D	4B	5A	6D	7C	8.2	P429
9.2	1C	2D	3B	4D	5A	6 (a) Con	7 (b) div		

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5. 求極限 → 發散/收斂

① $\lim_{n\to\infty} \frac{1}{n}$ (0)

② $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

③ L'Hôpital's Law $\lim_{x\to\infty} \frac{x^2}{2^x - 1}$

④ Squeeze Theorem: $\lim_{n\to\infty} \frac{\cos^2 n}{(1.1)^n}$

Exercises - Sequences and Series

Multiple Choice Questions

注意

1. $\sum_{n=1}^{\infty} \frac{(3)^{n+1}}{5^n} =$ $\sum_{n=1}^{\infty} 3\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)^{n-1} \Rightarrow \begin{cases} a = 3\cdot\frac{3}{5} = \frac{9}{5} \\ r = \frac{3}{5} \end{cases} \Rightarrow \sum_{n=1}^{\infty} a r^{n-1} = \frac{a}{1-r} = \frac{9/5}{\left(1-\frac{3}{5}\right)} = \frac{9}{2}$

(A) $\frac{3}{5}$ (B) $\frac{5}{2}$ (C) $\frac{9}{2}$ (D) The series diverges

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$f(1) = \sum_{n=1}^{\infty} (\tan 1)^n = \sum_{n=1}^{\infty} (\tan 1)(\tan 1)^{n-1}$

2. If $f(x) = \sum_{n=1}^{\infty} (\tan x)^n$, then $f(1) =$ $a = \tan 1 \quad r = \tan 1 > 1 \Rightarrow$ diverges

(A) -2.794 (B) -0.61 (C) 0.177 (D) The series diverges

$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \sum_{n=2}^{\infty} \frac{2}{(n-1)(n+1)} = \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1}\right) = \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\frac{1}{n-3} - \cancel{\frac{1}{n-1}}\right) + \left(\frac{1}{n-2} - \frac{1}{n}\right) + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right)$

3. $\sum_{n=2}^{\infty} \frac{2}{n^2-1} =$ $= \lim_{n\to\infty}\left(\frac{1}{1} + \frac{1}{2} - \frac{1}{n+1}\right) = \frac{3}{2}$

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) $\frac{3}{2}$

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4. The sum of the geometric series $\frac{2}{21} + \frac{4}{63} + \frac{8}{189} + \ldots$ is

(A) $\frac{5}{21}$ (B) $\frac{2}{7}$ (C) $\frac{4}{7}$ (D) The series diverges

$\frac{2}{21} + \frac{4}{63} + \frac{8}{189} + \cdots = \frac{2}{21}\left(1 + \frac{2}{3} + \frac{4}{9} + \cdots\right) = \frac{2}{21}\left[\left(\frac{2}{3}\right)^0 + \left(\frac{2}{3}\right)^1 + \left(\frac{2}{3}\right)^2 + \cdots\right]$

$a = \frac{2}{21} \quad r = \frac{2}{3} \qquad S = \frac{a}{1-r} = \frac{2/21}{\left(1-\frac{2}{3}\right)} = \frac{2}{21}\cdot 3 = \frac{2}{7}$

$S_n = \left(\frac{3^{n-1}}{3^n}\right)\left(\frac{7+n}{4+n}\right)^{20} = \frac{1}{3}\left(\frac{7+n}{4+n}\right)^{20} = \frac{1}{3}\left(\frac{3+(4+n)}{4+n}\right)^{20} = \frac{1}{3}\left(\frac{3}{4+n}+1\right)^{20}$

读题 → 5. If $S_n = \left(\frac{3^{n-1}}{(4+n)^{20}}\right)\left(\frac{(7+n)^{20}}{3^n}\right)$, to what number does the sequence $\{S_n\}$ converge?

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{1}{3}\left(\frac{3}{4+n}+1\right)^{20}\right] = \frac{1}{3}$

(A) $\frac{1}{3}$ (B) $\frac{7}{4}$ (C) $\left(\frac{7}{4}\right)^{20}$ (D) Diverges

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I: $(\cos^2 n) < 1 \Rightarrow I_n < \frac{1}{(1.1)^n} = \left(\frac{1}{1.1}\right)^n \Rightarrow$ Converge

6. Which of the following sequences converge?

II $\Rightarrow \left(\frac{e}{3}\right)^n - \left(\frac{1}{3}\right)^{n-1} \Rightarrow$ Converge

I. $\left\{\frac{\cos^2 n}{(1.1)^n}\right\}$ II. $\left\{\frac{e^n - 3}{3^n}\right\}$ III. $\left\{\frac{n}{9+\sqrt{n}}\right\}$

n 比 $9+\sqrt{n}$ 增长快 $\Rightarrow$ diverge

(A) I only (B) II only (C) III only (D) I and II only

I: $a_n = \frac{n}{10(n+1)} = \frac{1}{10}\left(\frac{n}{n+1}\right) = \frac{1}{10}\left(\frac{(n+1)-1}{n+1}\right) = \frac{1}{10}\left(1-\frac{1}{n+1}\right) \Rightarrow \lim_{n\to\infty} a_n = \frac{1}{10} \neq 0 \Rightarrow$ diverge

7. Which of the following series converge?

$a_n = \arctan n \Rightarrow \lim_{n\to\infty} a_n = \pm 1 \neq 0 \Rightarrow$ diverge

I. $\sum_{n=1}^{\infty} \frac{n}{10(n+1)}$ II. $\sum_{n=1}^{\infty} \arctan n$ III. $\sum_{n=1}^{\infty} \frac{-6}{(-5)^n} \Rightarrow a_n = (-6)\left(\frac{1}{-5}\right)^n = (-6)\left(\frac{1}{-5}\right)\left(\frac{1}{-5}\right)^{n-1}$

$a = \frac{6}{5}$, $r = -\frac{1}{5}$

几何级数

(A) I only (B) II only (C) III only (D) II and III only

Free Response Questions

8. Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+3)} + \frac{1}{7^n}\right)$.

$a_{n_1} = \frac{3}{n(n+3)} = \frac{3+n-n}{n(n+3)} = \frac{1}{n} - \frac{1}{n+3}$

$S_{n_1} = \left(\frac{1}{1} - \cancel{\frac{1}{4}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{5}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{7}}\right) + \dots$

$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{6+3+2}{6} = \frac{11}{6}$

$a_{n_2} = \frac{1}{7^n} = \frac{1}{7}\left(\frac{1}{7}\right)^{n-1} \Rightarrow \begin{cases} a = \frac{1}{7} \\ r = \frac{1}{7} \end{cases}$

$S_{n_2} = \frac{a}{1-r} = \frac{1}{7} \Big/ \left(1-\frac{1}{7}\right) = \frac{1}{7}\cdot\frac{7}{6} = \frac{1}{6}$

$S_{n_1} + S_{n_2} = \frac{11}{6} + \frac{1}{6} = 2$

1. The Integral test $\longleftarrow$ $a_n = f(n) \rightarrow$ 提 $\begin{cases} \text{positive} \\ \text{continuous} \\ \text{decreasing} \end{cases}$

if $\sum_{i=1}^{\infty} a_n \begin{cases} \text{is convergent} \\ \text{is dirvegent} \end{cases} \Longrightarrow \int_1^{\infty} f(x)\,dx \begin{cases} \text{is convergent} \\ \text{is dirvegent} \end{cases}$

例: Determine convergent/dirvergent

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

方法1: $\int_1^{\infty} \frac{1}{x^2+1}dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2+1}dx = \lim_{b\to\infty} \left[\arctan x\right]_1^b$

$= \lim_{b\to\infty} \left[\arctan b - \arctan 1\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

$\rightarrow$ Convergence

(b) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ $\quad \int_1^{\infty} \frac{\ln x}{x}$ $\left(u = \ln x \quad du = \frac{1}{x}\ln x,\ x=1 \quad u=0\right)$

$= \int_0^{\infty} du = u\Big|_0^{\infty}$

$\rightarrow$ divergence

2. P-Series and Harmonic Series (P級數與調合級數)

$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{n^p} + \cdots \begin{cases} p>1 \Rightarrow \text{Convergent} \\ 0<p\le 1 \Rightarrow \text{dirvergent} \end{cases}$

$p=1$: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \cdots$ harmonic series

Exercises - The Integral Test and *p*-Series

Multiple Choice Questions

1. If $\int_1^\infty \frac{dx}{x^2+1} = \frac{\pi}{4}$, then which of the following must be true?

I. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ diverges.

II. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

III. $\sum_{n=1}^{\infty} \frac{1}{n^2+1} = \frac{\pi}{4}$ ← 注意：Convergence 不意味着值相同

(A) none (B) I only (C) II only (D) II and III only

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2. What are all values of p for which $\int_1^\infty \frac{1}{\sqrt[3]{x^p}}dx$ converges?

$\int_1^\infty \frac{1}{\sqrt[3]{x^p}}dx = \int_1^\infty \frac{1}{x^{\frac{p}{3}}}dx \qquad \frac{p}{3} > 1 \Rightarrow p > 3$

(A) $P < -3$

(B) $P < -1$

(C) $P > 1$

(D) $P > 3$

(A) $u = 2x^2+1 \Rightarrow du = 4x,\ x=1\ u=3,\ \int_1^\infty \frac{x}{2x^2+1}dx = \frac{1}{4}\int_1^\infty \frac{4x}{2x^2+1}dx = \frac{1}{4}\int_3^\infty \frac{du}{u} = \frac{1}{4}\ln u\Big|_3^\infty \rightarrow$ dirvergence

3. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$ II. $\sum_{n=1}^{\infty} ne^{-n^2}$ III. $\sum_{n=2}^{\infty} \frac{1}{x\ln x}$

书上答案为(D)

但计算结果为(B)

(A) I only (B) II only (C) III only (D) I and II only

(B) $u = e^{-x^2} \Rightarrow du = -2xe^{-x^2}dx,\ x=1\ u=\frac{1}{e},\ \int_1^\infty xe^{-x^2}dx = \left(\frac{-1}{2}\right)\int_1^\infty (-2)xe^{-x^2}dx = \left(\frac{-1}{2}\right)\int_{\frac{1}{e}}^0 du = \left(\frac{-1}{2}\right)u\Big|_{\frac{1}{e}}^0 \rightarrow$ Convergence

(C) $u = \ln x,\ x=2\ u=\ln 2,\ \int_2^\infty \frac{1}{x\ln x}dx = \int_2^\infty \frac{\frac{1}{x}dx}{\ln x} = \int_2^\infty \frac{d(\ln x)}{\ln x} = \int_{\ln 2}^\infty \frac{du}{u} = \ln u\Big|_{\ln 2}^\infty \rightarrow$ divergence

4. What are all values of p for which $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^p+1}$ converges?

(A) $p>0$ (B) $p>\frac{1}{2}$ (C) $p>1$ (D) $p>\frac{3}{2}$

$\int_1^{\infty} \frac{\sqrt{x}}{x^p+1}\,dx = \int_1^{\infty} \frac{1}{x^{p-\frac{1}{2}}+\frac{1}{\sqrt{x}}}\,dx < \int_1^{\infty} \frac{1}{x^{p-\frac{1}{2}}}\,dx$, $p-\frac{1}{2}>1 \Rightarrow p>\frac{3}{2}$

5. What are all values of k for which the series $1+(\sqrt{2})^k+(\sqrt{3})^k+(\sqrt{4})^k+\cdots+(\sqrt{n})^k+\cdots$ converges?

$1+2^{\frac{k}{2}}+3^{\frac{k}{3}}+\cdots = \frac{1}{1}+\frac{1}{2^{-k/2}}+\frac{1}{3^{-k/2}}+\cdots$

(A) $k<-2$ (B) $k<-1$ (C) $k>1$ (D) $k>2$

$\frac{-k}{2}>1 \Rightarrow k<-2$

Free Response Questions

6. Determine whether the following series converge or diverge.

(a) $1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\cdots = \frac{1}{1}+\frac{1}{2^3}+\frac{1}{3^3}+\cdots$ $p=3>1 \Rightarrow$ Converges

(b) $1+\frac{1}{(\sqrt[3]{2})^2}+\frac{1}{(\sqrt[3]{3})^2}+\frac{1}{(\sqrt[3]{4})^2}+\cdots = \frac{1}{1}+\frac{1}{2^{\frac{2}{3}}}+\frac{1}{3^{\frac{2}{3}}}+\cdots$ $p=\frac{2}{3}<1 \Rightarrow$ diverges